Uniqueness of Optimal Control for the Relative Controllability of Neutral Integro-differential Systems in Banach Spaces with Distributed Delays in the Control

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ABSTRACT

A new method of approach is derived for the proof of the existence of an optimal control for Neutral Integrodifferential Systems in Banach Spaces with Distributed Delays in the Control, using the properties of optimal control energy function. The set functions upon which our studies hinged were extracted from the mild solution of the system. Use was also made of the unsymmetric Fubini theorem to establish the mild solution. Necessary and sufficient conditions for an admissible control to be an optimal control are established.

Keywords: Optimal control, uniqueness, controllability, controllability grammian, set functions.

INTRODUCTION:

The pioneering work of Vito Volterra on the integration of the differential equations of dynamics and partial differential dynamical systems published in 1884 gave vent to the conception of integral equations of the volterra type **(Oconnor and et al (2005);Oraekie(2014))**. It is equally observed in **Balachandran and Dauer (1989)**; **Oraekie (2017**) that the mixed initial boundary hyperbolic partial differential equation which arises in the study of Lossless transmission lines can be replaced by an associated neutral differential equation. This equivalence has been the basis of a number of investigations of the stability properties of distributed networks. **(see Balachandran and Dauer (1997) and Oraekie(2015)).**

The problem of controllability of linear and nonlinear systems represented by ordinary differential equation in finite dimensional space has been extensively studied. Many authors have extended the controllability concept to infinite dimensional systems in Banach spaces with bounded operators **(see Naito (1989))**. Naito has studied the controllability of semi linear systems. **Yamamoto and Park (1990)** discussed the same problem for parabolic equation with uniformly bounded nonlinear term. While **Chukwu and Lenhart (1991)** studied the controllability of nonlinear systems in abstract spaces. **Quinn and Carmichael (1984)** showed that the controllability problem in Banach space can be converted into a fixed pointed theorem problem for a single-valued mapping. **Balachandran (1996, 1998);Oraekie(2017)** had studied the controllability and local null controllability of Sobolve type integrodifferential systems and functional differential systems in Banach spaces by using Schauders fixed point theorem.

The purpose of this paper is to investigate the uniqueness of the optimal controllability of the abstract neutral functional integrodifferential systems with distributed delays in the control of the form.

$$\frac{d}{dt}[x(t) - g(t, x_t)] + A(t)x(t) = \int_0^t f(s, x_s)ds + \int_{-h}^{0}(d_s D(s,r)u(s+r) \qquad (1.1)$$

$$x(to) = x_0 = \emptyset \in B, \qquad t \in [0, a] = T$$

where B is the phase space, the state variable $x(.)$ takes values in Banach space X and the control function $u(.)$ is given in $L_2(J, U)$ $\left(where \left|u_j\right| \leq 1, \quad j = 1, 2, \ldots \ldots.\right)$, the Banach space of admissible control functions with u a Banach space. D is a bounded linear operator from U into X, the unbounded linear operators $-A$ generates an analytic semi group and $f, g \in JxB \to X$ are appropriate functions.

2.0 *Preliminaries and Definitions*

Throughout this work X will be a Banach space with norm $\|.\|$, $-A: D(A) \to X$

will be the infinitesimal generator of a compact analytic semigroup of uniformly bounded

linear operator $T(t)$. Let $0 \epsilon \rho(A)$, then it is possible to define the fractional power A^α,

for $0 < \alpha \leq 1$, as a closed linear operator on its domain $D(A^\alpha)$.

To study the system (1.1), we assume that the histories

$$x_t: (-\infty, 0) \to X, \qquad x_t(\theta) = x(t + \theta)$$

belong to some abstract phase space B, which is defined axiomatically.

Thus, B will be a linear space of functions mapping $(-\infty, 0]$ into X, endowed with a norm $\|.\|_B$.

Let us assume that B satisfies the following axioms:

(1). *If $x: [\alpha, \alpha + a] \to X, \quad a > 0$, is continuous on $[\alpha, \alpha + a]$ and $x_\alpha \in B$, then for*

every time $\tau \in [\alpha, \alpha + a]$ the following conditions hold:

(a) x_t *is in* B;

(b) $\|x(t)\| \leq k \|x_\tau\|_B$;

(c) $\left\| x_\tau \right\|_B \leq H(\tau-\alpha) sup\{\| x(s) \| : \alpha \leq s \leq \tau \} + M(\tau-\alpha) \left\| x_\alpha \right\|_B$.

$Here\ , k > 0\ is\ a\ constant, H, M: [0,\infty) \to [0,\infty)$.

$H\ is\ continuous\ and\ \ M\ is\ locally\ bounded, and\ K, H, M\ are\ independent\ of\ x(\tau)$.

(d) $For\ the\ function\ x\ (.)\ in\ (1), x_t\ is\ a\ B-valued\ continuous\ function\ on\ [\alpha, \alpha + a];$

$(2) The\ space\ B\ is\ a\ complete\ space.$

$Now\ we\ can\ give\ basic\ assumptions\ on\ the\ \ system\ (1.1).$

(i) $g: [0,a] x\ B \to x\ is\ a\ continuous\ function, and\ there\ exists\ a\ constant\ \lambda\ \in (0,1)\ and\ P, P_1 > 0, such\ that\ the\ function\ g\ is\ X_\lambda -\ valued\ and\ satisfies\ the\ lipschitz\ condition:$

$$\left\| A^\lambda g\ (t_1, \emptyset_1) - A^\lambda g\ (t_2, \emptyset_2) \right\| \leq P(|t_1 - t_2| + \|\emptyset_1 - \emptyset_2\|_B)$$

$For\ 0 \leq t_1, t_2 \leq a\ ;\ \emptyset_1, \emptyset_2 \in B, and\ the\ inequality$

$$\left\| A^\lambda\ g\ (t, \emptyset) \right\| \leq P_1\ (\|\emptyset\|_B + 1)\ holds\ for\ t \in J = [0,a], \emptyset \in B.$$

$(3)\ The\ function\ \ f: [0,a] xB \to X\ \ satisfies\ the\ following\ conditions:$

(i) $For\ each\ t \in J, the\ function\ \ f(t,.): B \to X\ \ is\ continuous\ and\ for\ each\ \ \emptyset \in B,$

$the\ function\ \ f(., \emptyset): J \to X\ is\ strongly\ measurable,$

(ii) $For\ each\ positive\ number\ n\ , there\ is\ a\ positive\ function\ \alpha_n \epsilon\ L_1([0,a])\ such\ that$

$$\sup_{\|\emptyset\|_B \leq n} \|f(t, \emptyset)\| \leq \alpha_n(t)$$

and

$$\liminf_{n \to \infty} \frac{1}{n} \int_0^a \int_0^t \alpha_n\ (s) ds\ \ dt = \gamma < \infty$$

$(4)\ The\ linear\ operator\ \ W\ \ from\ \ U\ into\ X\ \ is\ defined\ by$

$$Wu = \int_0^a T(t-s) \left[\int_{-h}^{0} d_S\ C(s,l) \right] u(s+l) ds$$

$and\ there\ exists\ a\ bounded\ invertible\ operator\ \ w^{-1}\ defined\ in\ L_2\ (J;\ U)/ker\ w.,$
$where\ \ C\ \ is\ a\ bounded\ linear\ operator.$

$\mathbf{2.01}$: $\boldsymbol{Variation\ of\ parameters}$

$The\ function\ x(.): (-\infty, a] \to X\ \ is\ a\ solution\ of\ system\ (1.1)\ if\ x_0 = \emptyset,$

$then\ the\ restriction\ of\ \ x(.) to\ the\ interval\ [0,a]\ \ is\ continuous\ and\ for\ each$

$0 \leq t \leq a, \qquad the\ function\ AT(t-s) g\ (s, x_s), s \in [0,t] is\ integrable\ and$

$the\ following\ integral\ equation\ is\ the\ required\ solution\ of\ system\ (1.1).$

$$x(t) = T(t)[\emptyset(0) - g(0,\emptyset)] + g(t, xt) - \int_0^t AT(t-s)g(s, x_s)ds$$

$$+ \int_0^t T(t-s)\left[\int_{-h}^0 (d_S\, C(s,l)u(s+l) + \int_0^s f(\tau, x_\tau)d\tau\right] ds, t \in J \quad \dots\dots \quad \dots .(2.1)$$

$$(2.1) \Longrightarrow x(t) = T(t)[\emptyset(0) - g(0,\emptyset)] + g(t, x_t) - \int_0^t AT(t-s)g(s,\ x_s\ )\ ds$$

$$+ \int_0^t T(t-s)\left[\int_{-h}^0 (d_S\, C(s,l)u(s+l)\right] ds + \int_0^t T(t-s)\left[\left[\int_0^s f(\tau, x\tau)d\tau\right]\right] ds \ \dots \ (2.2)$$

$The\ \ fourth\ term\ in\ the\ right-hand\ side\ of\ system\ (2.2) contains\ the\ values\ of\ the\ control\ u(t)\ for\ \ t < 0, as\ well\ as\ for\ t < 0.$

$The\ values\ of\ the\ control\ u(t)\ for\ t \in [0-h, 0]\ enter\ into\ the\ definition\ of\ the\ initial\ complete\ state\ z_{t_0}.$

$To\ separate\ them, the\ fourth\ term\ of\ system\ (2.2) must\ be\ transformed\ by\ changing\ the\ order\ of\ integration. Using\ the\ unsymmetric\ Fubini\ theorem, we\ have\ the\ following\ equalities:$

$$x(t) = T(t)[\emptyset(0) - g(0,\emptyset)] + g(t, x_t) - \int_0^t AT(t-s)g(s, x_s)ds$$

$$+ \int_{-h}^0 d_S\, C\left(\int_0^t T(t-s)C(s,l)u(s+l)ds\right) + \int_0^t T(t-s)\left[\int_0^s f(\tau, x_\tau)d\tau\right] ds \quad \dots\dots\dots\dots (2.3))$$

$$= T(t)[\emptyset(0) - g(0,\emptyset)] + g(t, xt) - \int_0^t AT(t-s)g(s, x_s)ds + \int_0^t T(t-s)\left[\int_0^s f(\tau, x_\tau)d\tau\right] ds$$

$$+ \int_{-h}^o d_S\, C\left(\int_{0+l}^{t+l} T(t-s)C(s-l,l)u(s-l+l)ds\right)$$

$$= T(t)[\emptyset(0) - g(0,\emptyset)] + g(t, x_t) - \int_0^t AT(t-s)g(s, x_s)ds + \int_0^t T(t-s)\int_0^s f(\tau, x_\tau)d\tau\ ds$$

$$+ \int_{-h}^0 d_S\, C\left(\int_{0+l}^0 T(t-s)C(s-l,l)u_0(s)ds\right)$$

$$+\int_{-h}^{0} d_S\, C\left(\int_{0}^{t+l} T(t-s)\, C(s-l,l)u(s)ds\right) \ldots\ldots\ldots\ldots \quad (2.4)$$

where dcs denotes that the integration is in the Lebesque − Stielties sense with respect to the variable s in the function $C(s,l)$.

Let us introduce the following notation:

$$\hat{C}(s,l) = \begin{cases} C(s,l) for\ s < t, l \in R \\ 0 \quad for \quad s > t,\ \ l \in R \end{cases} \quad \ldots\ldots \quad \ldots\ldots\ldots \quad \ldots\ldots \quad (2.5)$$

Hence $x(t)$ *can be expressed in the following form*:

$$x(t) = T(t)[\emptyset(0) - g(0,\emptyset)] + g(t,xt) - \int_{0}^{t} AT(t-s)g(s,x_s)ds$$

$$+\int_{0}^{t} T(t-s)\int_{0}^{s} f(\tau,x_\tau)d\tau\, ds \quad + \int_{-h}^{0} d_S\, C\left(\int_{0+l}^{0} T(t-s)C(s-l\,,l)u_0(s)ds\right)$$

$$+\int_{-h}^{0} d_S\, C\left(\int_{0}^{t} T(t-s)\hat{C}(s-l,l)u(s)ds\right) \ldots\ldots\ldots\ldots \quad (2.6)$$

Using again the unsymmetric Fubini theorem, the equality (2.6) *can be rewritten in more convenient form as follows*:

$$x(t) = T(t)[\emptyset(0) - g(0,\emptyset)] + g(t,x_t) - \int_{0}^{t} AT(t-s)g(s,x_s)ds$$

$$+\int_{0}^{t} T(t-s)\int_{0}^{s} f(\tau,x_\tau)d\tau\, ds \quad + \int_{-h}^{0} d_S\, C\left(\int_{l}^{0} T(t-s)Cs-l\,,l)u_0(s)ds\right)$$

$$+\int_{0}^{t}\left(\int_{-h}^{0} T(t-s)d_S\hat{C}(s-l,l)u(s)ds\right) \quad (2.7)$$

Now let us consider the solution $x(t)$ *of system* (1.1) *for* $t = t_1 = a$.

$$x(t) = T(t)[\emptyset(0) - g(0,\emptyset)] + g(t_1\,,x_{t_1}) - \int_{0}^{t_1} AT(t\,1-s)\, g(s,x_s)\, ds$$

$$+\int_0^t T(t-s)\int_0^s f(\tau, x_\tau)d\tau\, ds + \int_{-h}^0 d_S\, C\left(\int_{0+l}^0 T(t-s)C(s-l,l)u_0(s)ds\right)$$

$$+\int_0^{t_1}\left(\int_{-h}^0 T(t_1-s)d_S\hat{C}(s-l,l)u(s)ds\right) \qquad (2.8)$$

Consider system (2.7), *for brevity, let,*

$$\beta(t) = T(t)[\emptyset(0) - g(0,\emptyset)] + g(t,x_t) - \int_0^t AT(t-s)g(s,x_s)ds \quad \dots\dots\dots \qquad (2.9)$$

$$\mu(t) = \int_0^t T(t-s)\int_0^s f(\tau, x_\tau)d\tau\, ds + \int_{-h}^0 dc\, s(\int_{0+l}^0 T(t-s)C(s-l,l)u_0(s)ds \qquad (2.10)$$

$$z(t,s) = \int_{-h}^0 T(t-s)d_S\,\hat{C}(s-l,l) \quad \dots\dots.. \quad \dots\dots\dots\dots. \quad \dots\dots\dots. \qquad (2.11)$$

Substituting (2.9), (2.10), (2.11) *in* (2.7), *we have a precise variation of constant formula for system* (1.1) *as*

$$x(t,x_0,u) = \beta(t) + \mu(t) + \int_0^t z(t,s)u(s)ds \quad \dots. \quad \dots. \qquad (2.12)$$

Definition* 2.1** (Complete State***)
The complete state for system (1.1) *is given by the set* $g(t) = \{x, u_t\}$.

Definition* 2.2** (Relative Controllability***)
The system (1.1) *is said to be relatively controllable on* $[0,a]$ *if for every initial complete state* $g(0)$ *and* $x_1 \in X$, *there exists a control function* $u(t)$ *defined on* $[0,a]$ *such that the solution of system* (1.1) *satisfies* $x(t_1) = x_1$.

2.02: *Basic Set Functions and Properties*.
Definition* 2.3** (Reachable set***)
The reachable set for the system (1.1) *is given as*

$$R(t_0,0) = \{\left\{\int_0^{t_1}\left[\int_{-h}^0 T(t_1-s)\, d_S\,\hat{C}\,(s-l,l)]\right]u(s)ds\right\}$$

$$Where\ U = \{u \in L_2\,([0,a];X) : |u_j| \le .1\,; j = 1,2,\dots\dots.,m\}$$

Definition* 2.4** (Attainable set***)

The attainable set for the system (1.1) *is given as* :

$$A(t\,,0) = \{x(t\,,x_0\,,u) : u \in U\}, Where\ U = \{u \in L_2\ ([0\,,a];X) : |u_j| \ \le .1;\ j = 1,2,\dots\dots.,m\}$$

Definition 2.5 (Target set)
The target set for system (1.1) *denoted by* $G(t_1\,,0)$ *is given as*:

$$G(t_1\,,0) = \{x(t,x_0\,,u) : t_1 \ \ge \tau > 0\ for\ fixed\ \tau\ and\ \ u \in U$$

Definition 2.6 (Controllability grammian
The controllability grammian of the system (1.1) *is given as*:

$$W(t_1\,,0) = \int_0^{t_1} \left[\int_{-h}^{0} T(t_1 - s) d_S\ \hat{C}\ (s - l\,,l) \right] \left[\int_{-h}^{0} T(t_1 - s)\ d_S\ \hat{C}\ (s - l,l) \right]^T$$

Where T denotes matrix transpose.

2.03: *Relationship Between the Set Functions*

We shall first establish the relationship between the attainable set and the reachable set to enable us see that once a property has been proved for one set, then it is applicable to the other.
From equation (2.7),

$$A(t\,,0) = \eta(t) + \ R(t\,,0)]\ \ ,for\ \ \in U, t \in [0\,,a],$$

$$Where \quad \eta(t) = \ \beta(t) + \ \mu(t).$$

This means that the attainable set is the translation of the reachable set through $\eta \in X$.
Using the attainable set, therefore, it is easy to show that the set functions possess the properties of convexity, closeness and compactness.
Also, the set functions are continuous on $[o,\infty]$ *to the metric space of compact subject of* $X = E^n$. ***Chukwu* (1988) *and Gyori* (1982)** *give impetus for adaptation of the proofs of these properties for system* (1.1).

Definition 2.7 (Properness)
The system (1.1) *is proper in* $X = E^n$ *on* $[0\,,a]$ *if span* $R(t\,,0) = X = E^n$
i.e if

$$C^T \left[\int_{-h}^{0} T(t - s) d_S\ \ \hat{C}(s - l\,,l) \right] = 0 \quad a.e.$$

$a > o \Longrightarrow C = 0\,; C \in X = E^n$

3. *Main Results*

Here, a new method of approach is derived for the proof of the existence of optimal control.

Theorem 3.1

Consider the system (1.1) *given as* :

$$\frac{d}{dt}[x(t) - g(t, x_t)] + A(t)x(t) = \int_0^t f(s, x_s)ds + \int_{-h}^{0} (d_S C(s, l)u(s + l) \quad \dots\dots\dots..(3.1)$$

with its standing hypothesis.

Suppose u^* is the optimal control of the system (3.1), *then it is unique.*

Proof:

Let u^* and v^* be optimal controls for the system (3.1), *the u^* and v^* maximize*

$$C^T \int_0^t \left[\int_{-h}^{0} T(t - s)d_S\hat{C}(s - l, l) \right], \qquad for\ t \in [0, a], \qquad a > 0.$$

over all admissible controls $u \in U$, and so we have the inequality with u^* as the optimal control given below:

$$C^T \int_0^t \left[\int_{-h}^{0} T(t - s)d_S\hat{C}(s - l, l) \right] u(s)ds \le C^T \int_0^{t^*} \left[\int_{-h}^{0} T(t - s)d_S\hat{C}(s - l, l) \right] u^*(s)ds \qquad (3.2)$$

Also, usingv^* as the optimal control, we have

$$C^T \int_0^t \left[\int_{-h}^{0} T(t - s)d_S\hat{C}(s - l, l) \right] u(s)ds$$

$$\le C^T \int_0^{t^*} \left[\int_{-h}^{0} T(t - s)d_S\hat{C}(s - l, l) \right] u^*(s)ds \qquad (3.3)$$

Taking maximum of u over $[-1,1]$, *the range of definition of u^* in* (3.2)*and* (3.3), *we have the equation*

$$C^T \int_0^t \left[\int_{-h}^{0} T(t - s)d_S\hat{C}(s - l, l) \right] max|u(s)|ds, \quad for\ -1 \le s \le 1.$$

$$= C^T \int_0^{t^*} \left[\int_{-h}^{0} T(t - s)d_S\hat{C}(s - l, l) \right] u^*(s)ds, \quad for\ u^* \in U \qquad \dots\dots\dots\dots\dots\dots. \qquad (3.4)$$

Also

$$C^T \int_0^t \left[\int_{-h}^{0} T(t - s)d_S\hat{C}(s - l, l) \right] max|u(s)|ds, \quad for\ -1 \le s \le 1.$$

$$= C^T \int_0^{t^*} \left[\int_{-h}^{0} T(t-s) d_S \hat{C}(s-l\,,l) \right] v^*(s) ds, \;\; for\; v^* \in U \qquad \ldots\ldots\ldots\ldots\ldots\ldots. \qquad (3.5)$$

$for\; u, v^* \in U, \qquad v^*\; being\; optimal\; and\; -1 \le s \le 1.$

$Subtracting\; equation\;\; (3.5)\; from\;\; (3.4, we\; have$

$$0 = C^T \int_0^{t^*} \left[\int_{-h}^{0} T(t-s) d_S \hat{C}(s-l\,,l) \right] \{ u^*(s) - v^*(s) \}\, ds$$

$$(since\; system\; (3.1)\; is\; controllable\; or\; proper).$$

$$\Rightarrow\; u*(s) - v*(s) = 0$$

$$\Rightarrow\; u*(s) = v*(s)$$

$$This\; establishes\; that\; optimal\; control\; is\; unique.$$

4. CONCLUSION

In this work, Neutral Functional Integro-differential systems in Banach spaces with Distributed Delays in the control were presented for optimal controllability analysis.

We established that an admissible control energy function of Neutral Functional Integro-differential systems in Banach spaces with Distributed Delays in the control is optimal control energy function of the system if and only if it is unique

The establishment of this uniqueness property of an optimal control energy function provides a new approach to proving the existence of an optimal control energy function of our system of interest..

References

1. **J. J. Oconnor and E. F. Robertson (2005) ,** “Biography of Vito Volterra (1860-1940), Publication of the School of Mathematics and Statistics, University of St Andrew, Scotland
2. **ORAEKIE, P.A., (2014):** Uniqueness of Optimal Control of Linear

Time-Varying Systems with Delay in the Control**, Tropical Journal of Applied Natural Sciences Vol 1 No1 ANSU, ULI Page155-162.**

3. Balachandraand J.P.Dauer(1998), “Controllability of Sobolev-type Integrodifferential Systems in Banach spaces”, Journal of Mathematical Analysis and Applications, 217, pp 335-348.

4.K.Balachandranand J.P.Dauer(1997),”Asymptotic Null Controllability of Nonlinear Neutral Volterra Integrodifferential Systems”, Journal of Mathematics System, Vol, 7, No.2, pp 1-14.

5**. ORAEKIE, P.A., (2015):** Location of the Target Set on Semi linear Dynamical Systems with Multiple Delays in the Control**, REIKO INTERNATIONAL JOURNAL OF SCIENCE AND TECHNOLOGY RESEARCH, Vol.6. NO.2A SPECIAL EDITION, PAGE 62.**

6. K.Naito (1989), Approximate Controllability for Trajectories of Semi-linear Control Systems”, Journal of optimum Theory and Applications, 6, pp 57-65. **.**

7**. ORAEKIE, P.A., (2017):** The Form of Optimal Control of Semi linear Dynamical Systems with Multiple Delays in the Control, ***American Academic & Scholarly Research Journal Vol. 9, No 2, page 48 - 57.***

8. M.Yamamoto and J. Park (1990), “Controllability for Parabolic Equations with uniformly bounded non-linear terms”, Journal of optimization theory and Applications, 66, pp 515-532.

9. E.N.Chukwu and S.M Len hart (1991), “Controllability Questions for Nonlinear Systems in Abstract spaces’, Journal of optimization theory and Applications, 18, pp 437-462.

10.M.D.Quinn and N. Carmichael (1984-1985), “An Approach to Non-linear Control Problems using Fixed point methods, degree theory and Pseudo-inverses”, .Journal of Functional Analysis and Optimization, 7, pp 197-219.

11. **Balachandran (1996, 1998)** Nonlinear Functional Differential System in Banach Spaces” Journal of Optimal Theory and Application,88, pp 61-75. **.**

12. ORAEKIE, P.A., (2017): Form of Optimal Control of Nonlinear Infinitely Space of Neutral Functional Differential Systems with Distributed Delays in the Control, American Academic & Scholarly Research Journal, vol9, No6, pp52-61.

13. Y.Hino,S. Murakami and T. Naito(1991), “Functional Differential Equations With Infinite delay”, Lecture Notes in Maths, vol.1473 Springer-Vorlag, Berlin.

14. E.N.Chukwu(1988), “The Time Optimal Control Theory of Linear Differential Equations of Neutral Type”, Journal of Computer, Mathematics and Applications

vol.16 pp 851-866.

15. Gyori (1982), "Delay Differential and Integrodifferential Equations in Biological Compartmental Models" Systems Science (Wroclaw) Poland, 8 pp 7-187.

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