Null Controllability of Semi-Linear Differential Systems of NonLocal Initial Conditions with Distributed Delays in the Control in Banach Spaces

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ABSTRACT

In this work, a Semi linear Differential System of Non Local Initial Conditions with Distributed Delays in the Control in Banach spaces of the form

$$x^1(t) = Ax(t) + f\big(t, x(t)\big) + \int_{-h}^{0} [d_\theta H(t,\theta)]\, u(t+\theta)$$

$$x(0) + g(x) = x_0$$

is presented for controllability analysis. Necessary and Sufficient Conditions for the System to be null controllable are established . *Use is made of the Unsymmetric Fubini theorem and Schauders' fixed point theorem to etablish results . Conditions are also placed on the perturbation f which guaratee that if the linear control base system is proper and if the uncotrolled linear system is uniformly asymptotically stable, then the Semilinear Differential System is nullcontrollable with constraints .*

Keywords: null-controllability, semi-linear, distributed delays, nonlocal initial conditions, Banach spaces

1. INTRODUCTION

Controllability and Null Controllability of nonlinear systems represented by differential and Integrodifferential equations in Banach Spaces have been investigated extensively by many authors; **Balachandran, K. Anandhi (2004), Y.K.Chang, J.J. Nieto(2009), Oraekie,P.A(2017)**.A method is to transform the controllability problem into a fixed point problem for an appropriate operator in a function space. However, **Balachandran and Kim(2003)** pointed out that controllability results are only true for ordinary differential Systems in finite-dimensional spaces if the corresponding operator semi groups are compact.**Xue,X(2008)** studied the existence of integral solutions for a nonlinear differential equations with nonlocal initial conditions through Huasdorff measure of no compactness in the separable and uniformly smooth Banach spaces. In his work, **Xue, X (2008)** dropped the compactness of semi group. The semi group in his work is a contraction semi group satisfying equicontinuity, which is a special case of a strongly continuous semigroup.With respect to controllability, it is known from the of **Hermes and J.P. La Salle (1969)** that if the linear ordinary control system

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) \qquad (1)$$

Is proper and if the free system

$$\dot{x}(t) = A(t)x(t) \qquad (2)$$

Is uniformly asymptotically stable, then system (1) is null controllable with constraints. A similar result was obtained by **Chukwu(1980)** for the delay system of the form

$$\dot{x}(t) = L(t\,, x_t) + B(t)u(t) \; + f\big(t, x_t, u(t)\big) \qquad (3)$$

$$where\,, L(t,\phi) = \sum_{k=0}^{\infty} A_k(t)\phi(-t_k) + \int_{-r}^{0} A(t\,,s)\phi(s)ds.$$

Shinba (1985) studied the nonlinear infinite delay system of the form

$$\dot{x}(t) = L(t\,, x_t) + B(t)u(t) \; + \int_{-\infty}^{0} A(\theta)x(\theta)d\theta + f\big(t, x_t, u(t)\big) \qquad (4)$$

And showed that system (4) is Euclidean null controllable if the linear base system

$$\dot{x}(t) = L(t\,, x_t) + B(t)u(t) \qquad (5)$$

Is proper and the free system

$$\dot{x}(t) = L(t\,, x_t) + B(t)u(t) \; + \int_{-\infty}^{0} A(\theta)x(\theta)d\theta \qquad (6)$$

is uniformly asymptotically stable, provided that f satisfies some growth conditions.

Onwuatu(1993), studied the neutral systems with infinite delay of the form

$$\frac{d}{dt}D(t\,, x_t) = L(t\,, x_t) + B(t)u(t) \; + \int_{-\infty}^{0} A(\theta)x(t+\theta)d\theta + f\big(t, x_t, u(t)\big) \qquad (7)$$

$$x(t) = \phi(t);\; t \in (-\infty, 0]$$

$$where\,, L(t,\phi) = \sum_{k=0}^{\infty} A_k(t)\phi(-t_k) + \int_{-r}^{0} A(t\,,s)\phi(s)ds.$$

He developed sufficient computable criteria for the null controllability of system (7).
*While **Oraekie (2018)** studied the nonlinear infinite neutral systems with Multiple Delays in the Control of the form:*

$$\frac{\mathrm{d}}{\mathrm{dt}}\mathrm{D}(\mathrm{t}, \mathrm{x_t}) = \mathrm{L}(\mathrm{t}, \mathrm{x_t}) + \sum_{j=1}^{m} \mathrm{B_j}\, \mathrm{u}\big(\mathrm{t} - \mathrm{h_j}\big) + \int_{-\infty}^{0} \mathrm{A}(\theta)\mathrm{x}(\mathrm{t}+\theta)\, \mathrm{d}\theta + \mathrm{f}\big(\mathrm{t}, \mathrm{x_t}, \mathrm{u}(\mathrm{t})\big) \ldots\ldots..(8)$$

*He developed sufficient computable criteria for the null controllability of the system (8). His results extend those of **Hermes and Salle (1969)**, **Chukwu (1980)**, **Sinba (1985)** and **Onwuatu (1993)** to nonlinear infinite neutral systems with multiple delays in the control.*
In this paper, therefore, we consider the null controllability of the Semilinear Differential Systems of Nonlocal Initial Conditions with Distributed Delays in the Control in Banach Spaces of the form:

$$x^1(t) = Ax(t) + \int_{-h}^{0} [d_\theta H(t,\theta)]\, u(t+\theta) + f\big(t, x(t)\big) \qquad (9)$$

$$x(0) + g(x) = x_0$$

$with\ the\ main\ objective\ of\ investigating\ the\ null\ controllability\ of\ the\ system(8).$

$Here, the\ state\ x(.)\ takes\ value\ in\ a\ Banach\ Space\ X = R^n\ with\ the\ norm\ |.|\ ;$

$the\ operator\ A\ generates\ a\ strong\ continuous\ not\ necessarily\ compact, semigroup\ T(t)\ in\ X.$

$And\ the\ control\ function\ u(.) is\ given\ Lebsgue\ square\ integrable\ functions\ L_2(J, U);$

$there\ is\ a\ Banach\ Space\ of\ admissible\ control\ functions\ with\ U\ a\ Banach\ Space. H(t, \theta)\ is\ an$

$nxn\ matrix\ function\ continuous\ at\ t\ and\ of\ bounded\ variation\ in\ \theta\ on\ [-h, 0], h > 0$

$for\ each\ t\ [t_0, t_1]\ ; t_1 > t_0.$

$The\ functions\ f: JxX \to X\ , g: C(J, X) \to X\ are\ continuous. Here, x_0 = x(0)\ is\ a\ given$

$element\ in\ X\ , C(J, X)\ denotes\ the\ Banach\ space\ of\ continuous\ functions\ x(.) : J \to X$

$with\ the\ norm\ \|x\| = sup\{|x(t)|\ , t \in J\}.$

$The\ nonlocal\ initial\ condition\ is\ a\ generizeation\ of\ tthe\ classical\ initial\ condition, which$

$was\ motivated\ by\ physical\ phenomena. The\ pioneering\ work\ on\ nonlocal\ conditions\ is\ due\ to$

$\boldsymbol{Byszewski}\ (\mathbf{1991})\ following\ by\ \boldsymbol{Fu.X.Ezzinbi}(\mathbf{2003}).$

2. Preliminaries and Notations

Consider the following dynamical system(9) given as

$$x^1(t) = Ax(t) + \int_{-h}^{0} [d_\theta H(t, \theta)]\, u(t + \theta) + f\big(t, x(t)\big) \qquad (11)$$

$$x(0) + g(x) = x_0$$

$If\ T(t, t_0): B \to B\ , t > t_0\ is\ defined\ by\ T(t, t_0)\phi = x_t(t_0, \phi)\ and\ the\ solution\ x(t)\ of$

$system(11) with\ the\ initial\ complete\ state\ \boldsymbol{y_{t_0}} = \{\boldsymbol{x_0}\ , \boldsymbol{u_0}\}\ is\ of\ the\ following\ form$

$(\boldsymbol{see\ Klamka}(\mathbf{1978})\boldsymbol{as\ contained\ in\ Klamka}(\mathbf{1980})$:

$$x(t) = T(t)[x_0 - g(x)] + \int_{t_0}^{t} T(t - s) f(s\ , x(s)) ds$$
$$+ \int_{t_0}^{t} T(t - s) \int_{-h}^{0} [d_\theta H(t\ , \theta)]\, u(t + \theta) ds \qquad (12)$$

$Where\ T(t - s)\ is\ the\ state\ transition\ of\ the\ following\ linear\ homogeneous\ system$

$$x^1(t) = Ax(t) \qquad (13)$$

The third term in the right – hand side of system(12)contains the values of the control $u(t)$ for $t < t_0$, as well as for $t > t_0$. The values of the control $u(t)$ for $t \in [t_0 - h , t_0]$ enter into the definition of initial complete state $\mathbf{y}_{t_0}$. To separate them , the third term of system(12) must be transformed by changing the order of integration..Using the Unsymmetric Fubini theorem, we have the following equalities:

$$x(t) = T(t)[x_0 - g(x)] + \int_{t_0}^{t} T(t-s)f(s , x(s))ds + \int_{-h}^{0} d_{H_\theta}\left(\int_{t_0}^{t} T(t-s)\, H(l , \theta)u(l+\theta)dl\right) \qquad (14)$$

$$\Rightarrow x(t) = T(t)[x_0 - g(x)] + \int_{t_0}^{t} T(t-s)f\big(s , x(s)\big)ds + \int_{-h}^{0} d_{H_\theta}\left(\int_{t_0+\theta}^{t+\theta} T(t-s)\, H(l-\theta , \theta)u(l-\theta+\theta)dl\right) \qquad (15)$$

$$= T(t)[x_0 - g(x)] + \int_{t_0}^{t} T(t-s)f\big(s , x(s)\big)ds + \int_{-h}^{0} d_{H_\theta}\left(\int_{t_0+\theta}^{t+\theta} T(t-s)\, H(l-\theta , \theta)u(l)dl\right) \qquad (16)$$

$$\Rightarrow \; x(t) = T(t)[x_0 - g(x)] + \int_{t_0}^{t} T(t-s)f\big(s , x(s)\big)ds + \int_{-h}^{0} d_{H_\theta}\left(\int_{t_0+\theta}^{t_0} T(t-s)\, H(l-\theta , \theta)u_{t_0}(l)dl\right) + \int_{-h}^{0} d_{H_\theta}\left(\int_{t_0}^{t+\theta} T(t-s)\, H(l-\theta , \theta)u(l)dl\right) \qquad (17)$$

Where the symbol d_{H_θ} denotes that the integration is in the Lebesque – Sieltjes senes with respect to the variable θ in the function $H(l , \theta)$.

Let us introduce the following notation

$$H_t(l,\theta) = \begin{cases} H(l , \theta), & l < t, \theta \in R \\ 0, & l > t , \theta \in R \end{cases} \qquad (18)$$

Thus, $x(t)$ can be expressed in the following form:

$$\Rightarrow \; x(t) = T(t)[x_0 - g(x)] + \int_{t_0}^{t} T(t-s)f\big(s , x(s)\big)ds$$

$$+\int_{-h}^{0} d_{H_\theta}\left(\int_{t_0+\theta}^{t_0} T(t-s)\, H(l-\theta\,,\theta)u_{t_0}(l)dl\right)$$

$$+\int_{-h}^{0} d_{H_\theta}\left(\int_{t_0}^{t} T(t-s)\, H_t(l-\theta\,,\theta)u(l)dl\right) \qquad (19)$$

Using again the Unsymmetric Fubini theorem, the equality (19) *can be rewritten in a more convenient form as follows*:

$$x(t) = T(t)[x_0 - g(x)] + \int_{t_0}^{t} T(t-s)f\big(s\,,x(s)\big)ds$$

$$+\int_{-h}^{0} d_{H_\theta}\left(\int_{t_0+\theta}^{t_0} T(t-s)\, H(l-\theta\,,\theta)u_{t_0}(l)dl\right)$$

$$+\int_{t_0}^{t}\left(\int_{-h}^{0} T(t-s)\, d_\theta H_t(l-\theta\,,\theta)\right)u(l)dl \qquad (20)$$

Now let us consider the system (20) − *the exact mild solution of the system*(8) *for* $t = t_1$

$$x(t_1) = T(t_1)[x_0 - g(x)] + \int_{t_0}^{t_1} T(t_1-s)f\big(s\,,x(s)\big)ds$$

$$+\int_{-h}^{0} d_{H_\theta}\left(\int_{t_0+\theta}^{t_0} T(t_1-s)\, H(l-\theta\,,\theta)u_{t_0}(l)dl\right)$$

$$+\int_{t_0}^{t_1}\left(\int_{-h}^{0} T(t_1-s)\, d_\theta H_{t_1}(l-\theta\,,\theta)\right)u(l)dl \qquad (21)$$

2.1 BASIC SET FUNCTION AND PROPERTIES.

Definition 2.1.1 (Reachable Set)

The reachable set of the system(9)*denoted by* $R(t_1\,,t_0)$ *is given as*:

$$\boldsymbol{R(t_1\,,t_0)} = \left\{\int_{t_0}^{t_1}\left(\int_{-h}^{0} T(t_1-s)\, d_\theta H_{t_1}(l-\theta\,,\theta)\right)u(l)dl \;\; : u \in U;\, |u_j| \le 1\,;\, j = 1,2,\dots,m\right\}$$

where $U = \{u \in L_2([t_0\,,t_1]\,,R^m)\}$.

Definition 2.1.2 (Target Set)

The target set for the system(9)*denoted by* $G(t_1\,,t_0)$ *is given as*:

$G(t_1\,,t_0) = \{x(t_1\,,x_0\,,u) :\; t_1 \ge \tau > t_0\,, for\ some\ fixed\;\; \tau \in [t_0\,,t_1]\;\; and\ u \in U\}$.

Definition 2.1.3 (Attainable Set)

$The\ attainable\ set\ for\ the\ system (9)\ denoted\ by\ A(t_1\ ,t_0)\ is\ given\ as$:

$A(t_1\ ,t_0) = \{x(t_1\ ,x_0\ ,u) :\ u \in U\ ;\ |u_j| \leq 1\ ; j = 1,2,\ldots,m\ \};\ U = \{u \in L_2([t_0\ ,t_1]\ ,R^m)\}$.

Definition2.1.4 (Controllability Grammian or Map)

$The\ controllability\ grammian\ or\ map\ of\ the\ system (9) denoted\ by\ W(t_1\ ,t_0)\ is\ given\ as$

$$W(t_1\ ,t_0) = \int_{t_0}^{t_1} \left(\int_{-h}^{0} T(t_1 - s)\ d_\theta H_{t_1}(l - \theta\ ,\theta) \right) \left(\int_{-h}^{0} T(t_1 - s)\ d_\theta H_{t_1}(l - \theta\ ,\theta) \right)^T$$

Where T denotes matrix transpose.

$$\boldsymbol{If \quad Y(t_1) = \int_{-h}^{0} T(t_1 - s)\ d_\theta H_{t_1}(l - \theta\ ,\theta)} \qquad (22)$$

$$\boldsymbol{Then,\quad W(t_1\ ,t_0) = \int_{t_0}^{t_1} Y(t_1)\, Y^T(t_1)\ and \qquad W^{-1}(t_1\ ,t_0) = \frac{1}{\int_{t_0}^{t_1} Y(t_1)\, Y^T(t_1)}} \qquad (23)$$

Definition2.1.5 (Properness)

$The\ system\ (9) is\ said\ to\ be\ proper\ on\ an\ interval\ \ [t_0\ ,t_1]\ if$

$$C^T \int_{-h}^{0} T(t_1 - s)\ d_\theta H_{t_1}(l - \theta\ ,\theta) = 0\ ae\ , l \in [t_0\ ,t_1] \Rightarrow C = 0; C \in R^n.$$

$If\ the\ system (9) is\ proper\ on\ each\ interval\ [t_0\ ,t_1]\ ; t_1 > t_0\ , we\ say\ that\ system (9)\ is$ $proper\ in\ R^n$.

Definition2.1.6 (Positive Definite)

$The\ controllability\ grammian\ or\ map\ of\ the\ system (9) denoted\ by\ W(t_1\ ,t_0)\ is\ said\ to\ be$ $positive\ definite\ if\ W(t_1\ ,t_0)\ varnishes\ only\ at\ the\ origin\ and\ W(x) > 0\ ,$ $for\ all\ x \neq 0\ ; x \in D\ , where\ D = \{x \in R^n :\ \|x\| \leq r\ ; r > 0\}\ \subset R^n$.

Definition2.1.7 (Complete Controllability)

$The\ system (9)\ is\ said\ to\ be\ completely\ controllable\ on\ the\ interval\ [t_0\ ,t_1]\ if\ for\ every$ $function\ \phi\ and\ every\ state\ x_1 \in R^n, there\ exists\ an\ admissible\ control\ energy$ $function\ u \in U\ such\ that\ x(t_1) = x_1$.

Definition2.1.8 (Complete State)

$$We\ denote\ the\ complete\ state\ of\ system(9)\ by\quad \boldsymbol{z(t)} = \{\boldsymbol{x(t)}\ ,\boldsymbol{u_t}\}$$

$Then, the\ initial\ complete\ state\ of\ system(9)\ at\ time\ t_0\ is\ \boldsymbol{z(t_0)} = \{\boldsymbol{x(t_0)}\ ,\boldsymbol{u_{t_0}}\}$

Definition2.1.9 (Null Controllability)

$The\ system(9)\ is\ said\ to\ be\ null\ controllable\ on\ the\ interval\ [t_0\ ,t_1]\ if\ for\ every$

$function\ \phi \in B([t_0\ ,t_1], R^n)\ , there\ exists\ a\ time\ t_1 \geq t_0\ , u \in L_2([t_0\ ,t_1]\ ,P)\ ,P\ a\ compact$

$convex\ subset\ of\ R^m\ such\ that\ the\ solution\ x(t\ ,t_0\ ,\phi\ \ ,f)\ of\ system(9)\ satisfies$

$$x_{t_0}(t_0\ ,\phi\ \ ,f) = \phi\ \ and\ x(t_1\ ,t_0\ ,\phi\ \ ,f) = 0$$

Definition2.1.10 (Relative Controllability)

$$The\ system(9)\ is\ said\ to\ berelatively\ controllable\ on\ the\ interval\ [t_0\ ,t_1]\ if$$

$$A(t_1\ ,t_0) \cap G(t_1\ ,t_0) \neq \phi\ \ ,\qquad t_1\ > t_0 \in [t_0\ ,t_1].$$

3. MAIN WORK

The following theorems on controllability of system(9) are similar to the corresponding results for linear control systems of various types including some with delays and some without delays**(see Oraekie(2017),Onwuatu(1993),Hermes and La Salle(1963)).**

Theorem 3.1

The following statements are equivalent:

(i) $The\ controllability\ grammian\ W(t_1\ ,t_0)\ of\ sysem(9) is\ non-singular$

(ii). $System(9)\ is\ completely\ controllable\ \ on\ the\ interval\ [t_0\ ,t_1].\ t_1\ > t_0$

(iii). $System(9)\ is\ proper\ on\ the\ interval\ [t_0\ ,t_1].\ t_1\ > t_0$

Proof

$The\ controllability\ grammian\ W(t_1\ ,t_0)\ of\ sysem(9) is\ nonsingular\ is\ equivalent\ to\ saying$

$that\ it\ is\ positive\ definite, which\ in\ turn\ is\ equivalent\ to\ saying\ that\ the\ C^T of\ \ the\ controllability$

$index\ of\ system(9) is\ equal\ to\ zero\ almost\ everywhere\ , implies\ that\ C = 0.$

$$i.e\ .\ C^T \int_{-h}^{0} T(t_1 - s)\, d_\theta H_{t_1}(l-\theta\,,\theta) = 0\ a\ e\,, l \in [t_0\,, t_1] \Rightarrow C = 0; C \in R^n. Thus, showing\ that (i) and (iii) are\ \ equivalent..$$

$Now\ consider$

$$C^T \int_{t_0}^{t_1} \left(\int_{-h}^{0} T(t_1 - s)\, d_\theta H_{t_1}(l-\theta\,,\theta) \right) u(l)dl = 0\ ae\,, l \in [t_0\,, t_1].$$

$\boldsymbol{For\ each\ l, then}$

$$\int_{t_0}^{t_1} C^T \left(\int_{-h}^{0} T(t_1 - s)\, d_\theta H_{t_1}(l-\theta\,,\theta) \right) u(l)dl = \ C^T \left[\int_{t_0}^{t_1} \left(\int_{-h}^{0} T(t_1 - s)\, d_\theta H_{t_1}(l-\theta\,,\theta) \right) u(l)dl \right] = \mathbf{0}$$

$It\ follows\ from\ this\ \ that\ C\ is\ orthogonal\ to\ the\ reachable\ set\ \boldsymbol{R}(\boldsymbol{t_1}\,, \boldsymbol{t_0}).$

$If\ we\ assume\ the\ relative\ controllability\ of system(9)\ now, then$

$\boldsymbol{R}(\boldsymbol{t_1}\,, \boldsymbol{t_0}) = R^n\ , sothat\ C = 0\ \ .Showing\ that\ \ (ii)\ implies\ (iii).$

$Conversely, assume\ that\ system(9) is\ not\ controllable\ so\ that$

$$\boldsymbol{R}(\boldsymbol{t_1}\,, \boldsymbol{t_0}) = R^n, t_1\ > t_0\ .$$

$Then\ there\ exists\ \ C \neq 0, C \in R^n\ \ such\ that$

$$C^T \boldsymbol{R}(\boldsymbol{t_1}\,, \boldsymbol{t_0}) = 0$$

$It\ follows\ now\ that\ for\ all\ admissible\ controls\ u \in U \subset L_2([t_0\,, t_1], R^n)$

$$0 = \ C^T \left[\int_{t_0}^{t_1} \left(\int_{-h}^{0} T(t_1 - s)\, d_\theta H_{t_1}(l-\theta\,,\theta) \right) u(l)dl \right]$$

$$Hence, C^T \boldsymbol{R}(\boldsymbol{t_1}\,, \boldsymbol{t_0}) = 0\ \ \ a\ e\,, l \in [t_0\,, t_1], C \neq 0.$$

$This\ situation, impliesthat\ system(9)\ is\ not\ proper\ by\ the\ definition\ of\ properness\ since$ $C \neq 0. Hence, the\ system(9) is\ relatively\ controllable\ on\ [t_0\,, t_1]\ and\ hence\ completely$ $controllable.$

Theorem 3.2

$Assume\ for\ system\ (9)\ that:$

$(i).the\ constraint\ set\ U\ is\ an\ arbitrary\ compact\ subset\ of\ R^n.$

$(ii). the\ system(6)\ satisfies\ \ exponential\ estimate.$

$$i.e. \|x(t\ , t_0, \phi\ ,0\ )\| \leq Me^{-\delta(t-t_0)}\|\phi\|\ , \qquad for\ some\ \delta > 0, M > 0.$$

$(iii). the\ linear\ control\ system - \big(system(5)\big), is\ proper\ in\ R^n.$

$(iv). the\ continuous\ function\ f\ satisfies$

$|f(t\ , x(.), u(.))| \leq exp(-Nt)\pi\big(x(.), u(.)\big), for\ all\ \big(t\ , x(.), u(.)\big) \in [t_0\ , \infty)xExL_2\ ,$
$where\ \int_{t_0}^{\infty} \pi\big(x(s), u(s)\big)ds \leq \lambda < \infty\ and\ N - \delta \geq 0\ , then\ system(9) is\ null\ controllable.$

Proof

$By\ (iii) - the\ linear\ base\ control\ system\ \big(system(5)\big), there\ exists\ an\ inverse\ of$ $the\ controllability\ grammian\ say\ W^{-1}(\boldsymbol{t_1}\ , \boldsymbol{t_0})\ for\ each\ time\ t_1 > t_0.\ Suppose\ that\ the$ $pair\ of\ functions\ x\ and\ u\ form\ a\ solution\ pair\ to\ the\ set\ of\ integral\ equations$:

$$u(t) = -\left[\int_{-h}^{0} T(t_1 - s)\, d_\theta H_{t_1}(l - \theta\ , \theta)\right]^T W^{-1}(\boldsymbol{t_1}\ , \boldsymbol{t_0})\ \left[T(t_1)\big(x_0 - g(x)\big) + \int_{t_0}^{t_1} T(t_1 - s) f\big(s\ , x(s)\big)ds + \int_{-h}^{0} d_{H_\theta}\left(\int_{t_0+\theta}^{t_0} T(t_1 - s)\, H(l - \theta\ , \theta) u_{t_0}(l)dl\right)\right]$$

Substituting equation (22) and (23) into the above, we have

$$u(t) = \frac{-\boldsymbol{Y^T(t_1)}}{\int_{\boldsymbol{t_0}}^{\boldsymbol{t_1}} \boldsymbol{Y(t_1)\, Y^T(t_1)}}\left[T(t_1)\big(x_0 - g(x)\big) + \int_{t_0}^{t_1} T(t_1 - s) f\big(s\ , x(s)\big)ds + \int_{-h}^{0} d_{H_\theta}\left(\int_{t_0+\theta}^{t_0} T(t_1 - s)\, H(l - \theta\ , \theta) u_{t_0}(l)dl\right)\right] \qquad (24)$$

$$x(t) = T(t)[x_0 - g(x)] + \int_{t_0}^{t} T(t - s) f\big(s\ , x(s)\big)ds + \int_{-h}^{0} d_{H_\theta}\left(\int_{t_0+\theta}^{t_0} T(t - s)\, H(l - \theta\ , \theta) u_{t_0}(l)dl\right) + \int_{t_0}^{t}\left(\int_{-h}^{0} T(t - s)\, d_\theta H_t(l - \theta\ , \theta)\right) u(l)dl \qquad (25)$$

$x(t) = \phi(t), t \in [t_0 - \lambda, t_0].$

Then u is square integrable on $[t_0, t_1]$ *and x is a solution of system(9) corresponding to u with the initial state* $x(t_0) = \phi$.

Also, using u as expressed in equation (24), we have

$$x(t_1) = T(t_1)[x_0 - g(x)] + \int_{t_0}^{t_1} T(t_1 - s) f\big(s, x(s)\big) ds + \int_{-h}^{0} d_{H_\theta} \left(\int_{t_0+\theta}^{t_0} T(t_1 - s)\, H(l - \theta, \theta) u_{t_0}(l) dl \right)$$
$$+ \int_{t_0}^{t_1} \left(\int_{-h}^{0} T(t_1 - s)\, d_\theta H_{t_1}(l - \theta, \theta) \right) \frac{-\boldsymbol{Y^T(t_1)}}{\int_{t_0}^{t_1} \boldsymbol{Y(t_1)\, Y^T(t_1)}} \Bigg[T(t_1)\big(x_0 - g(x)\big) + \int_{t_0}^{t_1} T(t_1 - s) f\big(s, x(s)\big) ds + \int_{-h}^{0} d_{H_\theta} \left(\int_{t_0+\theta}^{t_0} T(t_1 - s)\, H(l - \theta, \theta) u_{t_0}(l) dl \right) \Bigg] \quad (26)$$

$$\textbf{But } \boldsymbol{Y(t_1)} = \int_{-h}^{0} \boldsymbol{T(t_1 - s)\, d_\theta H_{t_1}(l - \theta, \theta)}, \textbf{ therefore, we have}$$

$$x(t_1) = T(t_1)[x_0 - g(x)] + \int_{t_0}^{t_1} T(t_1 - s) f\big(s, x(s)\big) ds + \int_{-h}^{0} d_{H_\theta} \left(\int_{t_0+\theta}^{t_0} T(t_1 - s)\, H(l - \theta, \theta) u_{t_0}(l) dl \right)$$
$$+ \int_{t_0}^{t_1} \big(\boldsymbol{Y(t_1)}\big) \left(\frac{-\boldsymbol{Y^T(t_1)}}{\int_{t_0}^{t_1} \boldsymbol{Y(t_1)\, Y^T(t_1)}} \right) \Bigg[T(t_1)\big(x_0 - g(x)\big) + \int_{t_0}^{t_1} T(t_1 - s) f\big(s, x(s)\big) ds + \int_{-h}^{0} d_{H_\theta} \left(\int_{t_0+\theta}^{t_0} T(t_1 - s)\, H(l - \theta, \theta) u_{t_0}(l) dl \right) \Bigg] \quad (27)$$

$$\Rightarrow\ x(t_1) = T(t_1)[x_0 - g(x)] + \int_{t_0}^{t_1} T(t_1 - s) f\big(s, x(s)\big) ds + \int_{-h}^{0} d_{H_\theta} \left(\int_{t_0+\theta}^{t_0} T(t_1 - s)\, H(l - \theta, \theta) u_{t_0}(l) dl \right)$$
$$- \left(\frac{\int_{t_0}^{t_1} \boldsymbol{Y(t_1) Y^T(t_1)}}{\int_{t_0}^{t_1} \boldsymbol{Y(t_1)\, Y^T(t_1)}} \right) \Bigg[T(t_1)\big(x_0 - g(x)\big) + \int_{t_0}^{t_1} T(t_1 - s) f\big(s, x(s)\big) ds + \int_{-h}^{0} d_{H_\theta} \left(\int_{t_0+\theta}^{t_0} T(t_1 - s)\, H(l - \theta, \theta) u_{t_0}(l) dl \right) \Bigg] \quad (28)$$

$$\Rightarrow\ x(t_1) = T(t_1)[x_0 - g(x)] + \int_{t_0}^{t_1} T(t_1 - s) f\big(s, x(s)\big) ds$$

$$+\int_{-h}^{0} d_{H_\theta}\left(\int_{t_0+\theta}^{t_0} T(t_1-s)\,H(l-\theta\,,\theta)u_{t_0}(l)dl\right)$$
$$-\mathbf{1}\Bigg[T(t_1)\big(x_0-g(x)\big)+\int_{t_0}^{t_1} T(t_1-s)f\big(s\,,x(s)\big)ds$$
$$+\int_{-h}^{0} d_{H_\theta}\left(\int_{t_0+\theta}^{t_0} T(t_1-s)\,H(l-\theta\,,\theta)u_{t_0}(l)dl\right)\Bigg] \qquad (29)$$

$$\Rightarrow\ x(t_1)=T(t_1)[x_0-g(x)]+\int_{t_0}^{t_1} T(t_1-s)f\big(s\,,x(s)\big)ds$$
$$+\int_{-h}^{0} d_{H_\theta}\left(\int_{t_0+\theta}^{t_0} T(t_1-s)\,H(l-\theta\,,\theta)u_{t_0}(l)dl\right)$$
$$-\big[T(t_1)\big(x_0-g(x)\big)\big]-\int_{t_0}^{t_1} T(t_1-s)f\big(s\,,x(s)\big)ds$$
$$-\int_{-h}^{0} d_{H_\theta}\left(\int_{t_0+\theta}^{t_0} T(t_1-s)\,H(l-\theta\,,\theta)u_{t_0}(l)dl\right)=0\ .$$

It remains to show that the function $u: [t_0\,, t_1] \to U$ *is an admissible control. That is, we need to show that* $u: [t_0\,, t_1] \to U$ *is in the arbitrary compact constraint subset of* R^m. *That is* $|u| \le \delta_1$ *, for some constant* $\delta_1 > 0$.
By (ii) of theorem 3.2, we have

$$\left|\left[\int_{-h}^{0} T(t_1-s)\,d_\theta H_{t_1}(l-\theta\,,\theta)\right]^T W^{-1}(\boldsymbol{t_1}\,,\boldsymbol{t_0})\right|<\lambda_1$$

$$i.e\quad ;\ \left|\frac{\boldsymbol{Y^T(t_1)}}{\int_{\boldsymbol{t_0}}^{\boldsymbol{t_1}} \boldsymbol{Y(t_1)\,Y^T(t_1)}}\right|<\lambda_1\ ,\qquad \text{for some } \lambda_1>0 \ and$$

$$|\,T(t_1)[x_0-g(x)]|\le\lambda_2\exp\big(-\delta(t_1-t_0)\big)\quad ,\text{for some constant } \lambda_2>0$$

Hence,

$$|u(t)|\le\lambda_1\big[\lambda_2 exp\big(-\delta(t_1-t_0)\big)\big]\int_{t_0}^{t_1}\lambda_3 exp\big[-\delta(t_1-s)exp(-Ns)\pi\big(x(.),u(.)\big)ds\big]$$

Thus,

$$|u(t)|\le\lambda_1\big[\lambda_2 exp\big(-\delta(t_1-t_0)\big)\big]+\lambda\lambda_3 exp(-\delta t_1) \qquad (30)$$

since $N-\delta\ge 0$ *and* $s\ge t_0\ge 0$.

Hence, by taking t sufficiently large, we have

$|u(t)|\le\delta_1$, $t\in[t_0\,,t_1]$ *, showing that u is an admissible control.*

Finally, we now prove the existence of a solution pair of the integral equations (24) *and* (25).

Let E *be the Banach space of all functions* $(x,u)\colon [t_0-h\,,t_1]x[t_0-h\,,t_1] \to R^n x R^m$,

where $x \in \boldsymbol{E}([\boldsymbol{t_0}-\boldsymbol{h}\,,\boldsymbol{t_1}]\,,\boldsymbol{R^n})\,;u \in \boldsymbol{L_2}([\boldsymbol{t_0}-\boldsymbol{h}\,,\boldsymbol{t_1}]\,,\boldsymbol{R^m})$ *with the norm defined by*

$$\|(x\,,u)\| = \|x\|_2 + \|u\|_2$$

$$where\,,\|x\|_2 = \sqrt{\int_{t_0-h}^{t_1} |x(s)|^2\,ds}\,;\quad ,\|u\|_2 = \sqrt{\int_{t_0-h}^{t_1} |u(s)|^2\,ds}$$

We define the operator T *by* $T\colon E \to E$ *by* $T(x\,,u) = (y\,,v)$ *, where*

$$v(t_1) = \frac{-\boldsymbol{Y^T(t_1)}}{\int_{\boldsymbol{t_0}}^{\boldsymbol{t_1}} \boldsymbol{Y(t_1)}\,\boldsymbol{Y^T(t_1)}}\Bigg[T(t_1)\big(x_0 - g(x)\big) + \int_{t_0}^{t_1} T(t_1-s)f\big(s\,,x(s)\big)ds + \int_{-h}^{0} d_{H_\theta}\left(\int_{t_0+\theta}^{t_0} T(t_1-s)\,H(l-\theta\,,\theta)u_{t_0}(l)dl\right)\Bigg] \qquad (24)$$

And $v(t) = \omega(t)$ *, for some* $t \in [t_0-\lambda\,,t_0]$

$$y(t_1) = T(t_1)[x_0 - g(x)] + \int_{t_0}^{t_1} T(t_1-s)f\big(s\,,x(s)\big)ds + \int_{-h}^{0} d_{H_\theta}\left(\int_{t_0+\theta}^{t_0} T(t_1-s)\,H(l-\theta\,,\theta)u_{t_0}(l)dl\right) + \int_{t_0}^{t_1}\left(\int_{-h}^{0} T(t_1-s)\,d_\theta H_{t_1}(l-\theta\,,\theta)\right)u(l)dl \qquad (25)$$

$y(t) = \phi(t), t \in [t_0-\lambda\,,t_0]$.

We have already shown that $|u(t)| \le \delta_1$ *,* $t \in J = [t_0\,,t_1]$ *and also for the function*

$$v:\ [t_0-h\,,t_0] \to U\,,$$

we have $|v(t)| \le \delta_1$ *. Hence,* $\|x\|_2 \le \delta_1 (t_0+h-t_0)^{\frac{1}{2}} = N_0$.

Again, $|y(t)| \le \lambda_2 exp[-\delta(t_1-t_0)] + \lambda_4 \int_{t_0}^{t_1} |v(s)|\,ds + \lambda\lambda_3 exp(-\delta t_1)$, *where*

$$\lambda_4 = sup\left|\int_{-h}^{0} T(t_1-s)\,d_\theta H_{t_1}(l-\theta\,,\theta)\right|.$$

Since $\delta > 0\,, t_1 \ge t_0 \ge 0$ *, we deduce that*

$$|y(t)| \le \lambda_2 + \lambda_4\delta(t_1-t_0) + \lambda\lambda_3 = N_1 \quad , t \in [t_0\,,t_1].$$

$$and\ \ , |y(t)| \leq \sup|\phi| = \beta \, , t \in [t_0 - r \, , t_0].$$

Hence, if $M = \max[N_1, \beta]$, then , $\|y\|_2 \leq M(t_0 + h - t_0)^{\frac{1}{2}} = N_2 < \infty$.

Let $\rho = \max[N_0 \, , N_2]$.

Then , if $G(\rho) = \{(x \, , u) \in E : \|x\|_2 \leq \rho \, , \|u\|_2 \leq \rho \}$, we have thus shown that the

operator T maps G into its self. i.e, $T : G(\rho) \to G(\rho)$.

Since $G(\rho)$ is closed, bounded and convex, by ***Riesz theorem*** *as contained in* **Kantorovica , L. V and G. P. Akilov (1982),** ***p297***, *and* ***Oraekie*****(2017)** . ***Onwuatu*** **(1993)** *it is relatively compact under the transformation of T . Hence, the Schauders' fixed point theorem implies that T has a fixed point. Thus, the system(9) is null controllable.*

4 . *CONCLUTION*

The Set Functions upon which our studies hinged are also extracted from the mild solution which we cultivated. Necessary and Sufficient Conditions for the null controllability of the Semi linear differential systems with distributed delays in the control have been derived.

These conditions are given with respect to the controllability of the linear controlled base system of system(9) and the uniformly asymptotic stability of the uncontrolled linear system of the system(9),assuming that the perturbation f satisfies some smoothness and growth conditions. These results extended known results in the literature.

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